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Perfect 4-colorings of some generalized Peterson graph

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Abstract

The notion of a perfect coloring, introduced by Delsarte, generalizes the concept of completely regular code. A perfect z-colorings of a graph is a partition of its vertex set. It splits vertices into z parts $P_1, \cdots, P_z$ such that for all $i, j \in \{1, \cdots, z\}$, each vertex of P_i is adjacent to p_{ij}, vertices of P_j. The matrix $P = (p_{ij})_{i,j \in \{1,\cdots,z\}}$, is called parameter matrix. In this article, we classify all the realizable parameter matrices of perfect 4-colorings of some the generalized peterson graph.

Keywords: Parameter matrices, Perfect coloring, Equitable partition, Generalized peterson graph.

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1. Introduction

The concept of a perfect z-coloring plays a significant role in graph theory, algebraic combinatorics, and coding theory (completely regular codes). There is another phrase for this concept in the writing as "equitable partition" see [8]. In 1973, Delsarte conjectured the non-existence of nontrivial perfect codes in Johnson graphs. Since then, some effort has been made to count the parameter matrices of some Johnson graphs, including $J(4, 2)$, $J(5, 2)$, $J(6, 2)$, $J(6, 3)$, $J(7, 3)$, $J(8, 3)$, $J(8, 4)$, and $J(v, 3)$ (v odd) [2, 3, 7].

Fon-Der-Flass count the parameter matrices (perfect 2-colorings) of n-dimensional hypercube Q_n for $n < 24$. He also obtained some constructions and a necessary condition for the existence of perfect 2-colorings of the n-dimensional cube with a given parameter matrix [4, 5, 6]. In this article, we classify the parameter matrices of all perefect 4-colorings of some generalized peterson graph.

Some generalized peterson graph including $GP(7, 1)$, $GP(8, 1)$, $GP(8, 2)$ and $GP(8, 3)$ given as follow:

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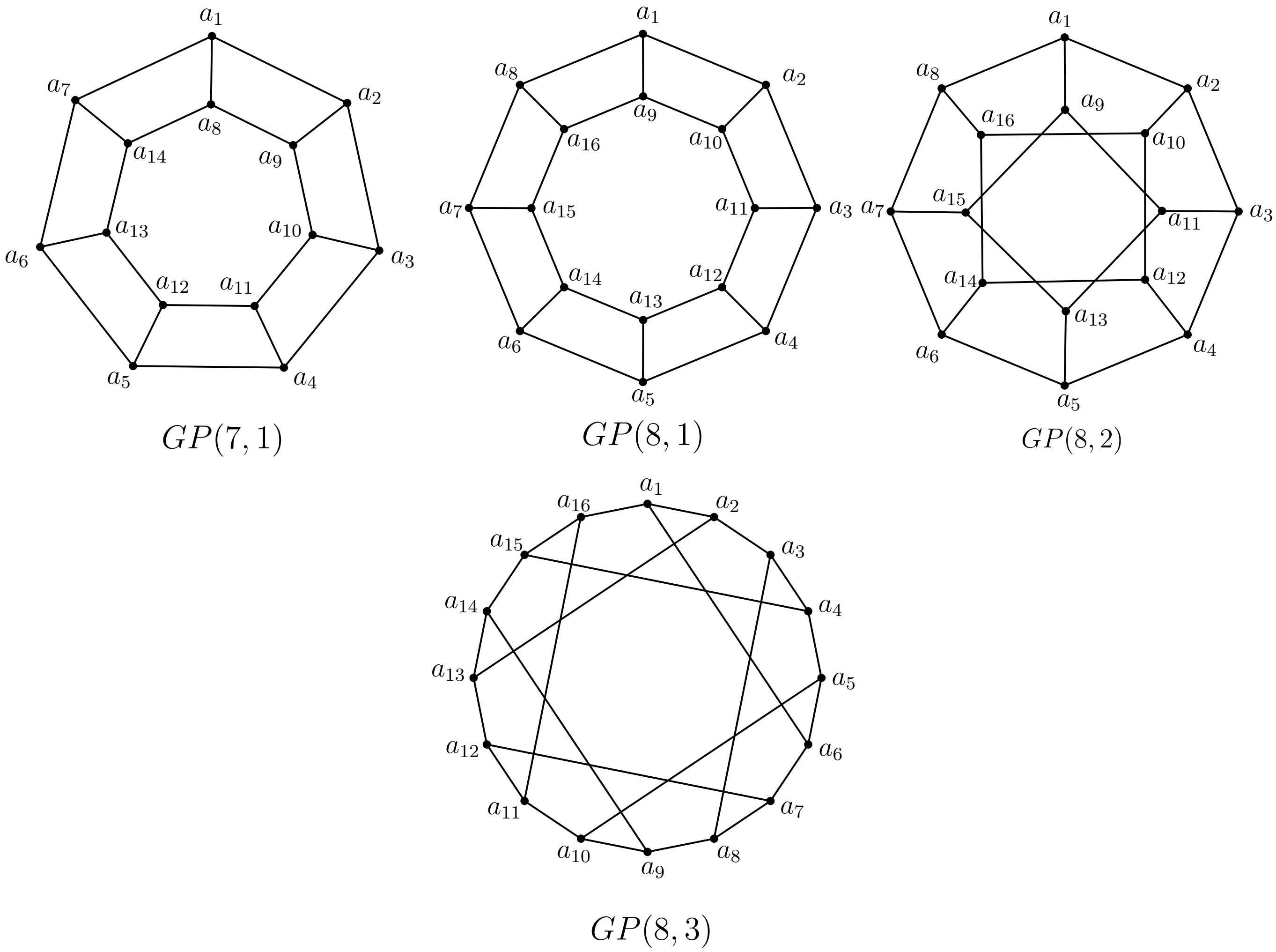

Figure 1: Some generalized peterson graph

Definition 1.1. The generalized peterson graph $\mathsf{GP}(n,k)$ has vertices,respectively, edges given by

$$V\big(GP(n,k)\big) = \{a_i, b_i : 0 \leqslant i \leqslant n-1\},$$
$$E\big(GP(n,k)\big) = \{a_i a_{i+1}, a_i b_i, b_i b_{i+k} : 0 \leqslant i \leqslant n-1\},$$

Where the subscripts are expressed as integers modulo n ($\geqslant 5$) , and k ($\geqslant 1$) is the skip.

Definition 1.2. For a graph G and an integer z, a mapping $T : V(G) \longrightarrow \{1,2,\cdots,z\}$ is called a perfect z-coloring with matrix $P = (p_{ij})_{i,j\in\{1,\cdots,z\}}$, if it is surjective,and for all i,j,for every vertex of color i, the number of its neighbours of color j is equal to p_{ij} . The matrix P is called the parameter matrix of a perfect coloring. In the case $z = 4$, we call the first color white that show by W, the second color black that show by B and the third color red that show by R and the color foure green that show by G.

2. Preliminaries

In this section, we present some results concerning necessary conditions for the existence of perfect 4-coloring of some generalized peterson graph with a given parameter matrix

$$P = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$$

The simplest necessary condition for the existence of perfect 4-colorings of some generalized peterson with the matrix $\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$ is

$$a + b + c + d = e + f + g + h = i + j + k + l = m + n + o + p = 4.$$

Theorem 2.1. *[8] If* T *is a perfect coloring of a graph* G *with* z *colors, then any eigenvalue of* T *is an eigenvalue of* G.

Theorem 2.2. *[1] Let* T *a perfect 4-coloring of a graph* G *with matrix* $P = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$

(1) if $b, c, d \neq 0$, *then*

$$|W| = \frac{|V(G)|}{1 + \frac{b}{e} + \frac{c}{i} + \frac{d}{m}}, \qquad |B| = \frac{|V(G)|}{\frac{e}{b} + 1 + \frac{ec}{bi} + \frac{ed}{bm}},$$

$$|R| = \frac{|V(G)|}{\frac{i}{c} + \frac{ib}{ce} + 1 + \frac{id}{cm}}, \qquad |G| = \frac{|V(G)|}{\frac{m}{d} + \frac{mb}{de} + \frac{mc}{di} + 1}.$$

(2) if $b, c, h \neq 0$, *then*

$$|W| = \frac{|V(G)|}{1 + \frac{b}{c} + \frac{c}{i} + \frac{bh}{en}}, \qquad |B| = \frac{|V(G)|}{\frac{e}{b} + 1 + \frac{ec}{bi} + \frac{h}{n}},$$

$$|R| = \frac{|V(G)|}{\frac{i}{c} + \frac{ib}{ce} + 1 + \frac{ibh}{cen}}, \qquad |G| = \frac{|V(G)|}{\frac{ne}{hb} + \frac{n}{h} + \frac{nec}{hbi} + 1}.$$

(3) if $b, c, l \neq 0$, *then*

$$|W| = \frac{|V(G)|}{1 + \frac{b}{e} + \frac{c}{i} + \frac{cl}{io}}, \qquad |B| = \frac{|V(G)|}{\frac{e}{b} + 1 + \frac{ec}{bi} + \frac{ecl}{bio}},$$

$$|R| = \frac{|V(G)|}{\frac{i}{c} + \frac{ib}{ce} + 1 + \frac{l}{o}}, \qquad |G| = \frac{|V(G)|}{\frac{oi}{lc} + \frac{oib}{lce} + \frac{o}{l} + 1}.$$

(4) *if* $b, d, g \neq 0$, *then*

$$|W| = \frac{|V(G)|}{1 + \frac{b}{e} + \frac{bg}{ej} + \frac{d}{m}}, \qquad |B| = \frac{|V(G)|}{\frac{e}{b} + 1 + \frac{e}{j} + \frac{ed}{bm}},$$

$$|R| = \frac{|V(G)|}{\frac{je}{gb} + \frac{j}{g} + 1 + \frac{jeb}{gbm}}, \qquad |G| = \frac{|V(G)|}{\frac{m}{d} + \frac{mb}{de} + \frac{mbg}{dej} + 1}.$$

(5) *if* $b, d, l \neq 0$, *then*

$$|W| = \frac{|V(G)|}{1 + \frac{b}{e} + \frac{do}{ml} + \frac{d}{m}}, \qquad |B| = \frac{|V(G)|}{\frac{e}{b} + 1 + \frac{edo}{bml} + \frac{ed}{bm}},$$

$$|R| = \frac{|V(G)|}{\frac{lm}{od} + \frac{lmb}{ode} + 1 + \frac{l}{o}}, \qquad |G| = \frac{|V(G)|}{\frac{m}{d} + \frac{mb}{de} + \frac{o}{l} + 1}.$$

(6) *if* $b, g, h \neq 0$, *then*

$$|W| = \frac{|V(G)|}{1 + \frac{b}{e} + \frac{bg}{ej} + \frac{bh}{en}}, \qquad |B| = \frac{|V(G)|}{\frac{e}{b} + 1 + \frac{g}{j} + \frac{h}{n}},$$

$$|R| = \frac{|V(G)|}{\frac{je}{gb} + \frac{j}{g} + 1 + \frac{jh}{gn}}, \qquad |G| = \frac{|V(G)|}{\frac{ne}{hb} + \frac{n}{h} + \frac{ng}{hj} + 1}.$$

(7) *if* $b, g, l \neq 0$, *then*

$$|W| = \frac{|V(G)|}{1 + \frac{b}{e} + \frac{bg}{ej} + \frac{bgl}{ejo}}, \qquad |B| = \frac{|V(G)|}{\frac{e}{b} + 1 + \frac{g}{j} + \frac{gl}{jo}},$$

$$|R| = \frac{|V(G)|}{\frac{je}{gb} + \frac{j}{b} + 1 + \frac{l}{o}}, \qquad |G| = \frac{|V(G)|}{\frac{oje}{lgb} + \frac{oj}{lg} + \frac{o}{l} + 1}.$$

(8) *if* $b, h, l \neq 0$, *then*

$$|W| = \frac{|V(G)|}{1 + \frac{b}{e} + \frac{bho}{enl} + \frac{bh}{en}}, \qquad |B| = \frac{|V(G)|}{\frac{e}{b} + 1 + \frac{ho}{nl} + \frac{h}{n}},$$

$$|R| = \frac{|V(G)|}{\frac{lne}{ohb} + \frac{ln}{oh} + 1 + \frac{l}{o}}, \qquad |G| = \frac{|V(G)|}{\frac{ne}{hb} + \frac{n}{h} + \frac{o}{l} + 1}.$$

(9) *if* $c, d, g \neq 0$, *then*

$$|W| = \frac{|V(G)|}{1 + \frac{cj}{ig} + \frac{c}{i} + \frac{d}{m}}, \qquad |B| = \frac{|V(G)|}{\frac{gi}{cj} + 1 + \frac{g}{j} + \frac{gid}{jcm}},$$

$$|R| = \frac{|V(G)|}{\frac{i}{c} + \frac{j}{g} + 1 + \frac{id}{cm}}, \qquad |G| = \frac{|V(G)|}{\frac{m}{d} + \frac{mcj}{dig} + \frac{mc}{di} + 1}.$$

(10) if $c, d, h \neq 0$*, then*

$$|W| = \frac{|V(G)|}{1 + \frac{dn}{mh} + \frac{c}{i} + \frac{d}{m}}, \qquad |B| = \frac{|V(G)|}{\frac{hm}{dn} + 1 + \frac{hmc}{ndi} + \frac{h}{n}},$$

$$|R| = \frac{|V(G)|}{\frac{i}{c} + \frac{idn}{cmh} + 1 + \frac{id}{cm}}, \qquad |G| = \frac{|V(G)|}{\frac{m}{d} + \frac{n}{h} + \frac{mc}{di} + 1}.$$

(11) if $c, g, h \neq 0$*, then*

$$|W| = \frac{|V(G)|}{1 + \frac{cj}{ig} + \frac{c}{i} + \frac{cjh}{igh}}, \qquad |B| = \frac{|V(G)|}{\frac{gi}{jc} + 1 + \frac{g}{j} + \frac{h}{n}},$$

$$|R| = \frac{|V(G)|}{\frac{i}{c} + \frac{j}{g} + 1 + \frac{jh}{gn}}, \qquad |G| = \frac{|V(G)|}{\frac{ngi}{hjc} + \frac{n}{h} + \frac{ng}{hj} + 1}.$$

(12) if $c, g, l \neq 0$*, then*

$$|W| = \frac{|V(G)|}{1 + \frac{cj}{ig} + \frac{c}{i} + \frac{cl}{io}}, \qquad |B| = \frac{|V(G)|}{\frac{gi}{jc} + 1 + \frac{g}{j} + \frac{gl}{jo}},$$

$$|R| = \frac{|V(G)|}{\frac{i}{c} + \frac{j}{g} + 1 + \frac{l}{o}}, \qquad |G| = \frac{|V(G)|}{\frac{oi}{lc} + \frac{oj}{lg} + \frac{o}{l} + 1}.$$

(13) if $c, h, l \neq 0$*, then*

$$|W| = \frac{|V(G)|}{1 + \frac{cln}{ioh} + \frac{c}{i} + \frac{cl}{io}}, \qquad |B| = \frac{|V(G)|}{\frac{hoi}{nlc} + 1 + \frac{ho}{nl} + \frac{h}{n}},$$

$$|R| = \frac{|V(G)|}{\frac{i}{c} + \frac{ln}{oh} + 1 + \frac{l}{o}}, \qquad |G| = \frac{|V(G)|}{\frac{oi}{lc} + \frac{n}{h} + \frac{o}{l} + 1}.$$

(14) if $d, g, h \neq 0$*, then*

$$|W| = \frac{|V(G)|}{1 + \frac{dn}{mh} + \frac{dng}{mhj} + \frac{d}{m}}, \qquad |B| = \frac{|V(G)|}{\frac{hm}{nd} + 1 + \frac{g}{j} + \frac{h}{n}},$$

$$|R| = \frac{|V(G)|}{\frac{jhm}{gnd} + \frac{j}{g} + 1 + \frac{jh}{gn}}, \qquad |G| = \frac{|V(G)|}{\frac{m}{d} + \frac{n}{h} + \frac{ng}{hj} + 1}.$$

(15) if $d, g, l \neq 0$*, then*

$$|W| = \frac{|V(G)|}{1 + \frac{doj}{mlg} + \frac{do}{ml} + \frac{d}{m}}, \qquad |B| = \frac{|V(G)|}{\frac{glm}{jod} + 1 + \frac{g}{j} + \frac{gl}{jo}},$$

$$|R| = \frac{|V(G)|}{\frac{lm}{od} + \frac{j}{g} + 1 + \frac{l}{o}}, \qquad |G| = \frac{|V(G)|}{\frac{m}{d} + \frac{oj}{lg} + \frac{o}{l} + 1}.$$

(16) if $d, h, l \neq 0$, *then*

$$|W| = \frac{|V(G)|}{1 + \frac{dn}{mh} + \frac{do}{ml} + \frac{d}{m}}, \qquad |B| = \frac{|V(G)|}{\frac{hm}{nd} + 1 + \frac{ho}{nl} + \frac{h}{n}},$$

$$|R| = \frac{|V(G)|}{\frac{lm}{od} + \frac{ln}{oh} + 1 + \frac{l}{o}}, \qquad |G| = \frac{|V(G)|}{\frac{m}{d} + \frac{n}{h} + \frac{o}{l} + 1}.$$

Remark 2.3. The distinct eigenvalues of the graph $GP(7,1)$ are the numbers 3,1, The distinct eigenvalues of the graph $GP(8,1)$ are the numbers 3,1,-1, The distinct eigenvalues of the graph $GP(8,2)$ are the numbers 1,3 and the distinct eigenvalues of the graph $GP(8,3)$ are the numbers 3,1,-1.

By using Theorem 2.1, we only have the following matrices, which we have shown with $P_1, \cdots, P_{31}$.

$$P_1 = \begin{bmatrix} 0&0&0&3\\0&0&1&2\\0&1&0&2\\1&1&1&0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0&0&0&3\\0&0&1&2\\0&1&0&2\\1&1&1&0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0&0&0&3\\0&0&2&1\\0&2&1&0\\2&1&0&0 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 0&0&0&3\\0&0&3&0\\0&1&1&1\\1&0&1&1 \end{bmatrix}, \quad P_5 = \begin{bmatrix} 0&0&0&3\\0&1&0&2\\0&0&1&2\\1&1&1&0 \end{bmatrix},$$

$$P_6 = \begin{bmatrix} 0&0&0&3\\0&1&0&2\\0&0&2&1\\1&1&1&0 \end{bmatrix}, \quad P_7 = \begin{bmatrix} 0&0&0&3\\0&1&1&1\\0&1&2&0\\2&1&0&0 \end{bmatrix}, \quad P_8 = \begin{bmatrix} 0&0&0&3\\0&1&2&0\\0&1&0&2\\1&0&2&0 \end{bmatrix}, \quad P_9 = \begin{bmatrix} 0&0&1&2\\0&1&0&2\\1&0&2&0\\1&1&0&1 \end{bmatrix}, \quad P_{10} = \begin{bmatrix} 0&0&1&2\\0&1&1&1\\1&1&1&0\\2&1&0&0 \end{bmatrix},$$

$$P_{11} = \begin{bmatrix} 0&0&1&2\\0&1&2&0\\1&2&0&0\\2&0&0&1 \end{bmatrix}, \quad P_{12} = \begin{bmatrix} 0&0&1&2\\0&2&0&1\\1&0&2&0\\2&1&0&0 \end{bmatrix}, \quad P_{13} = \begin{bmatrix} 0&0&1&2\\0&2&1&0\\1&1&1&0\\2&0&0&1 \end{bmatrix}, \quad P_{14} = \begin{bmatrix} 0&0&3&0\\0&0&0&3\\1&0&0&2\\0&1&1&1 \end{bmatrix}, \quad P_{15} = \begin{bmatrix} 0&0&3&0\\0&0&1&2\\1&1&1&0\\0&1&0&2 \end{bmatrix},$$

$$P_{16} = \begin{bmatrix} 0&0&3&0\\0&0&1&2\\2&1&0&0\\0&1&0&2 \end{bmatrix}, \quad P_{17} = \begin{bmatrix} 0&0&3&0\\0&1&0&2\\1&0&0&2\\0&1&1&1 \end{bmatrix}, \quad P_{18} = \begin{bmatrix} 0&1&0&2\\1&0&0&2\\0&0&2&1\\1&1&1&0 \end{bmatrix}, \quad P_{19} = \begin{bmatrix} 0&1&0&2\\1&2&0&0\\0&0&0&3\\1&0&2&0 \end{bmatrix}, \quad P_{20} = \begin{bmatrix} 0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0 \end{bmatrix},$$

$$P_{21} = \begin{bmatrix} 0&1&1&1\\1&0&1&1\\1&1&1&0\\1&1&0&1 \end{bmatrix}, \quad P_{22} = \begin{bmatrix} 0&1&2&0\\1&2&0&0\\1&0&0&2\\0&0&1&2 \end{bmatrix}, \quad P_{23} = \begin{bmatrix} 0&3&0&0\\1&0&0&2\\0&0&1&2\\0&1&2&0 \end{bmatrix}, \quad P_{24} = \begin{bmatrix} 0&3&0&0\\1&0&0&2\\0&0&2&1\\0&1&1&1 \end{bmatrix}, \quad P_{25} = \begin{bmatrix} 0&3&0&0\\1&0&2&0\\0&1&0&2\\0&0&1&2 \end{bmatrix},$$

$$P_{26} = \begin{bmatrix} 1&0&0&2\\0&0&2&1\\0&2&1&0\\1&1&0&1 \end{bmatrix}, \quad P_{27} = \begin{bmatrix} 1&0&0&2\\0&1&0&2\\0&0&2&1\\1&1&1&0 \end{bmatrix}, \quad P_{28} = \begin{bmatrix} 1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1 \end{bmatrix}, \quad P_{29} = \begin{bmatrix} 1&0&1&1\\0&2&0&1\\1&0&2&0\\1&1&0&1 \end{bmatrix}, \quad P_{30} = \begin{bmatrix} 1&2&0&0\\1&0&0&2\\0&0&0&3\\0&1&2&0 \end{bmatrix},$$

$$P_{31} = \begin{bmatrix} 1&2&0&0\\1&0&0&2\\0&0&1&2\\0&1&2&0 \end{bmatrix}.$$

3. Perfect 4-colorings of some generalized peterson graph

The parameter matrices of $GP(7,1)$, $GP(8,1)$, $GP(8,2)$ and $GP(8,3)$ graphs are enumerated in the next teorems.

Theorem 3.1. *The graph* $GP(7,1)$ *has no perfect 4-colorings.*

Proof. A parameter matrix corresponding to perfect 4-colorings of the graph $GP(7,1)$ may be one of the matrices $P_1, \cdots, P_{31}$. By using Theorem 2.2, only the matrices P_1, P_{16}, P_{26} can be a parameter matrices, because the number of white, black, red and green are an integer. For matrix P_1, each vertex with color green has one adjacent vertices with color green. Now have the following possibilities:

(1) $T(a_1) = B$, $T(a_2) = T(a_3) = T(a_4) = T(a_5) = T(a_9) = R$, $T(a_6) = T(a_7) = T(a_8) = T(a_{13}) = G$, $T(a_{14}) = W$ and then $T(a_{11}) = G$, $T(a_{10}) = T(a_{12}) = B$, which is a contradiction with four row of matrix P_1.

(2) $T(a_1) = W$, $T(a_3) = T(a_{14}) = B$, $T(a_4) = T(a_5) = T(a_{11}) = T(a_{12}) = T(a_{13}) = R$ and $T(a_6) = T(a_7) = T(a_{10}) = G$ then $T(a_2) = T(a_8) = T(a_9) = G$,which is a contradiction with the four row of matrix P_1. Hence graph $GP(7,1)$ has no perfect 4-colorings with matrix P_1.

Similar to matrix P_1, we can proof for matrices P_{16} and P_{26} as follows:

For matrix P_{16}, each vertex with color white has three adjacent vertices with color red. Now have the following possibilities:

(3) $T(a_1) = T(a_2) = T(a_9) = T(a_{10}) = G$, $T(a_4) = T(a_6) = T(a_{12}) = R$, $T(a_3) = T(a_8) = B$ and $T(a_5) = T(a_{11}) = T(a_{13}) = W$ then $T(a_{14}) = R$ and $T(a_7) = G$, which is a contradiction with the three row of matrix P_{16}.

(4) $T(a_1) = T(a_5) = T(a_9) = T(a_{11}) = T(a_{13}) = W$, $T(a_3) = B$, $T(a_2) = T(a_4) = T(a_6) = T(a_{10}) = T(a_{12}) = R$ then $T(a_7) = T(a_8) = R$ and $T(a_{14}) = B$, which is a contradiction with the three row of matrix P_{16}. Hence graph $GP(7,1)$ has no perfect 4-colorings with matrix P_{16}.

For matrix P_{26}, each vertex with color white has two adjacent vertices with color green, and each vertex with color green has zero adjacent vertices with color red. Now have the following possibilities:

(5) $T(a_1) = T(a_3) = T(a_{12}) = T(a_{14}) = B$, $T(a_4) = T(a_5) = T(a_6) = T(a_7) = T(a_{13}) = R$, $T(a_8) = T(a_{10}) = T(a_{11}) = G$ then $T(a_2) = R$ and $T(a_9) = W$, which is a contradiction with the one row of matrix P_{26}.

(6) $T(a_1) = T(a_2) = T(a_3) = T(a_{10}) = T(a_{11}) = T(a_{14}) = R$, $T(a_4) = T(a_7) = T(a_8) = T(a_{12}) = B$, $T(a_5) = T(a_9) = G$ then $T(a_6) = G$ and $T(a_{13}) = R$, which is a contradiction with the four row of matrix P_{26}. Hence graph $GP(7,1)$ has no perfect 4-colorings with matrix P_{26}.

□

Theorem 3.2. *The graph* $GP(8,1)$ *has a perfect 4-colorings only with the matrices* P_{10}, P_{20}, P_{21} *and* P_{28}.

Proof. A parameter matrix corresponding to perfect 4-colorings of the graph $GP(8,1)$ may be one of the matrices $P_1, \cdots, P_{31}$. Using the Theorem 2.2, only the matrices P_4, P_{10}, P_{12}, P_{13}, P_{19}, P_{20}, P_{21}, P_{22}, P_{23}, P_{24}, and P_{28} can be a parameter matrices, because the number of white, black, red and green are an integer. For matrix P_4, each vertex with color white has three adjacent vertices with color green and each vertex with color red has one adjacent vertices with color green. Now have the following possibilities:

(1) $T(a_1) = W$, $T(a_4) = B, T(a_3) = T(a_5) = T(a_{11}) = T(a_{12}) = R$, $T(a_2) = T(a_7) = T(a_8) = T(a_9) = T(a_{10}) = T(a_{13}) = G$ then $T(a_{14}) = B$ and $T(a_{15}) = W$ and $T(a_{16}) = R$, which is a contradiction with one row of the matrix P_4.

(2) $T(a_1) = T(a_2) = T(a_6) = T(a_9) = T(a_{11}) = T(a_{14}) = G, T(a_3) = T(a_5) = T(a_{12}) = T(a_{13}) = R$, $T(a_7) = T(a_{10}) = W$, $T(a_4) = B$ then $T(a_8) = T(a_{15}) = G$ and $T(a_{16}) = R$, which is a contradiction with three row of the matrix P_4. Hence graph $GP(8,1)$ has no perfect 4- colorings with the matrix P_4.

The proof of the matrices P_{12}, P_{13}, P_{19}, P_{22}, P_{23}, P_{24} is similar to the proof of the matrix P_4. Consider the mapping T_1, T_2, T_3 and T_4 as follows:

$$T_1(a_1) = T_1(a_6) = T_1(a_{10}) = T_1(a_{13}) = W, \quad T_1(a_3) = T_1(a_4) = T_1(a_{15}) = T_1(a_{16}) = B$$
$$T_1(a_7) = T_1(a_8) = T_1(a_{11}) = T_1(a_{12}) = R, \quad T_1(a_2) = T_1(a_5) = T_1(a_9) = T_1(a_{14}) = G.$$

$$T_2(a_1) = T_2(a_5) = T_2(a_{11}) = T_2(a_{15}) = W, \quad T_2(a_2) = T_2(a_6) = T_2(a_{12}) = T_2(a_{16}) = B,$$
$$T_2(a_4) = T_2(a_8) = T_2(a_{10}) = T_2(a_{14}) = R, \quad T_2(a_3) = T_2(a_7) = T_2(a_9) = T_2(a_{13}) = G.$$

$$T_3(a_1) = T_3(a_5) = T_3(a_{11}) = T_3(a_{15}) = W, \quad T_3(a_2) = T_3(a_6) = T_3(a_{12}) = T_3(a_{16}) = B,$$
$$T_3(a_9) = T_3(a_{10}) = T_3(a_{13}) = T_3(a_{14}) = R, \quad T_3(a_3) = T_3(a_4) = T_3(a_7) = T_3(a_8) = G.$$

$$T_4(a_1) = T_4(a_4) = T_4(a_5) = T_4(a_8) = W, \quad T_4(a_{10}) = T_4(a_{11}) = T_4(a_{14}) = T_4(a_{15}) = B,$$
$$T_4(a_2) = T_4(a_3) = T_4(a_6) = T_4(a_7) = R, \quad T_4(a_9) = T_4(a_{12}) = T_4(a_{13}) = T_4(a_4) = G.$$

It is clear that T_1, T_2, T_3 and T_4 are perfect 4-coloring with the matrices P_{10}, P_{20}, P_{21} and P_{28} respectively. □

Theorem 3.3. *The graph* $GP(8, 2)$ *has a perfect 4-colorings with only the matrices* P_{10} *and* P_{12}.

Proof. A parameter matrix corresponding to perfect 4-colorings of the graph GP(8, 2) may be one of the matrices $P_1, \cdots, P_{31}$. By using Theorem 2.2, graph $GP(8, 2)$ can have perfect 4-colorings only with matrices P_{10}, P_{12}, P_{13}, P_{19}, P_{22} and P_{24}, because the number of white, black, red and green are an integer. For matrix P_{13}, each vertex with color white has one adjacent vertices with color red and two adjacent vertices with color green. Now have the following possibilities:

(1) $T(a_1) = T(a_4) = T(a_{10}) = T(a_{15}) = W$, $T(a_2) = T(a_3) = T(a_9) = T(a_{11}) = T(a_{12}) = T(a_{13}) = G, T(a_7) = T(a_8) = R$, $T(a_{14}) = T(a_{16}) = B$, then $T(a_5) = W$ and $T(a_6) = B$, which is a contradiction with one row of the matrix P_{13}.

(2) $T(a_1) = T(a_7) = T(a_8) = T(a_9) = T(a_{15}) = B$, $T(a_3) = T(a_5) = T(a_{14}) = W$, $T(a_4) = T(a_6) = T(a_{12}) = G$, $T(a_{11}) = T(a_{13}) = R$, then $T(a_2) = T(a_{16}) = R$ and $T(a_{10}) = W$, which is a contradiction with one row of the matrix P_{13}. Hence graph $GP(8, 2)$ has no perfect 4-colorings with the matrix P_{13}.

The proof of the matrices P_{19}, P_{22}, P_{24} is similar to the proof of the matrix P_{13}. Consider the mapping T_1 and T_2 as follows:

$$T_1(a_{11}) = T_1(a_{12}) = T_1(a_{15}) = T_1(a_{16}) = W, \quad T_1(a_1) = T_1(a_2) = T_1(a_5) = T_1(a_6) = B,$$
$$T_1(a_3) = T_1(a_4) = T_1(a_7) = T_1(a_8) = R, \quad T_1(a_9) = T_1(a_{10}) = T_1(a_{13}) = T_1(a_{14}) = G.$$

$$T_2(a_1) = T_2(a_3) = T_2(a_5) = T_2(a_7) = W, \quad T_2(a_{10}) = T_2(a_{12}) = T_2(a_{14}) = T_2(a_{16}) = B,$$
$$T_2(a_9) = T_2(a_{11}) = T_2(a_{13}) = T_2(a_{15}) = R, \quad T_2(a_2) = T_2(a_4) = T_2(a_6) = T_2(a_8) = G.$$

It is clear that T_1 and T_2 are perfect 4-coloring with the matrices P_{10} and P_{12} respectively. □

Theorem 3.4. *The graph* $GP(8, 3)$ *has a perfect 4-colorings only with the matrices* P_{20}, P_{21} *and* P_{28}.

Proof. A parameter matrix corresponding to perfect 4-colorings of the graph $GP(8, 3)$ may be one of the matrices $P_1, \cdots, P_{31}$. By using Theorem 2.2, graph $GP(8, 3)$ can have perfect 4- colorings with matrices P_{10}, P_{11}, P_{12}, P_{13}, P_{19}, P_{20}, P_{21}, P_{22}, P_{23}, P_{24} and P_{28}, because the number of white, black, red and green are an integer. For matrix P_{10}, each vertex with color white has one adjacent vertices with color red and two adjacent vertices with color green. Now have the following possibilities:

(1) $T(a_1) = T(a_6) = T(a_8) = T(a_9) = B$, $T(a_2) = T(a_3) = T(a_5) = T(a_{10}) = R$, $T(a_7) = T(a_{12}) = T(a_{14}) = T(a_{16}) = G$, $T(a_{11}) = T(a_{13}) = W$, then $T(a_4) = T(a_{15}) = W$, which is a contradiction with one row of the matrix P_{10}.

(2) $T(a_1) = T(a_5) = T(a_{16}) = R$, $T(a_2) = T(a_{11}) = W$, $T(a_3) = T(a_{10}) = T(a_{12}) = T(a_{13}) = T(a_{14}) = G$, $T(a_4) = T(a_6) = T(a_{15}) = B$, then $T(a_7) = T(a_8) = T(a_9) = W$, which is a contradiction with one row of the matrix P_{10}. Hence graph $GP(8,3)$ has no perfect 4-colorings with the matrix P_{10}.

The proof of the matrices $P_{11}, P_{12}, P_{13}, P_{19}, P_{20}, P_{23}, P_{24}$ is similar to the proof of the matrix P_{10}.
Consider the mapping T_1, T_2 and T_3 as follows :

$$T_1(a_1) = T_1(a_4) = T_1(a_9) = T_1(a_{12}) = W, \quad T_1(a_3) = T_1(a_6) = T_1(a_{11}) = T_1(a_{14}) = B,$$
$$T_1(a_5) = T_1(a_8) = T_1(a_{13}) = T_1(a_{16}) = R, \quad T_1(a_2) = T_1(a_7) = T_1(a_{10}) = T_1(a_{15}) = G.$$

$$T_2(a_1) = T_2(a_4) = T_2(a_9) = T_2(a_{12}) = W, \quad T_2(a_5) = T_2(a_8) = T_2(a_{12}) = T_2(a_{16}) = B,$$
$$T_2(a_2) = T_2(a_3) = T_2(a_{10}) = T_2(a_{11}) = R, \quad T_2(a_6) = T_2(a_7) = T_2(a_{14}) = T_2(a_{15}) = G.$$

$$T_3(a_1) = T_3(a_2) = T_3(a_9) = T_3(a_{10}) = W, \quad T_3(a_4) = T_3(a_7) = T_3(a_{12}) = T_3(a_{15}) = B,$$
$$T_3(a_3) = T_3(a_8) = T_3(a_{11}) = T_3(a_{16}) = R, \quad T_3(a_5) = T_3(a_6) = T_3(a_{13}) = T_3(a_{14}) = G.$$

It is clear that T_1, T_2 and T_3 are perfect 4-coloring with the matrices P_{20}, P_{21} and P_{28} respectively. □

Finally, we summarize the results of this paper in the following table.

Table 1: Parameter matrices of some generalized peterson graph

Graphs	Parameter Matrices
GP(7,1)	×
GP(8,1)	$P_{10}, P_{20}, P_{21}, P_{28}$
GP(8,2)	P_{10}, P_{12}
GP(8,3)	P_{20}, P_{21}, P_{28}

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